One important consideration is that the electric charge is invariant under Lorentz transformation. This means that $\rho$ must transform as

$$\rho' = \gamma \rho \qquad (53)$$

since the invariance of 4-dimensional volume, $d^4x' = \det\Lambda\, d^4x = d^4x$ implies that

$$d^3x' = \frac{\partial t}{\partial t'} d^3x = (\Lambda^{-1})^0{}_0\, d^3x = \frac{1}{\gamma} d^3x .$$

Then, if we take for instance, the Gauss law for the electric field, we get (using Gaussian system of units)

$$\partial'_x E'_x + \partial'_y E'_y + \partial'_z E'_z = 4\pi \rho'$$

$$\partial'_1 = (\Lambda^{-1})^\nu{}_1\, \partial_\nu = (\Lambda^{-1})^0{}_1\, \partial_0 + (\Lambda^{-1})^1{}_1\, \partial_1$$

$$= \frac{\gamma v}{c} \partial_0 + \gamma \partial_1$$

$$\frac{\gamma v}{c} \partial_0 E'_x + \gamma \partial_x E'_x + \partial_y E'_y + \partial_z E'_z = 4\pi \gamma \rho'$$

This is clearly quite different than the original equation. So, we conclude that the components of the fields must somehow change if we want

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to recover the Lorentz symmetry.

Notice that among other things, the time derivative of $E'_x$ should not appear in the Gauss law.

On the other hand, from the Ampère-Maxwell law we get, for its x component,

$$\partial'_y B'_z - \partial'_z B'_y = \frac{4\pi}{c} j'_x + \partial'_0 E'_x$$

From the continuity equation we get

$$\partial'_t \rho' + \partial'_x j'_x + \partial'_y j'_y + \partial'_z j'_z = 0$$

$$\partial'_0 = (\Lambda^{-1})^\mu{}_0 \partial_\mu = (\Lambda^{-1})^0{}_0 \partial_0 + (\Lambda^{-1})^i{}_0 \partial_i$$

$$\partial'_0 = \gamma \partial_0 + \gamma \frac{v}{c} \partial_x \quad \therefore \quad \partial'_t = \gamma \partial_t + \gamma v \partial_x$$

Then, setting $j'_y = j_y$ and $j'_z = j_z$ we have

$$\gamma \partial_t \rho' + \gamma v \partial_x \rho' + \gamma \partial_x j'_x + \frac{\gamma v}{c^2} \partial_t j'_x + \partial_y j_y + \partial_z j_z = 0$$

$$\partial_t \left( \gamma \rho' + \frac{\gamma v}{c^2} j'_x \right) + \partial_x \left( \gamma v \rho' + \gamma j'_x \right) + \partial_y j_y + \partial_z j_z = 0$$

and since we have $\rho' = \gamma\rho$, we then find

$$\begin{cases} j_x = \gamma^2 v \rho + \gamma j'_x \\ \rho = \gamma^2 \rho + \dfrac{\gamma v}{c^2} j'_x \end{cases}$$

; From this 2nd equation

$$\rho(1-\gamma^2) = \frac{\gamma v}{c^2} j'_x \; ; \; \rho\left(1 - \frac{1}{1-v^2/c^2}\right) = \frac{\gamma v}{c^2} j'_x$$

$$-\frac{\gamma^2 v^2}{c^2}\rho = \frac{\gamma v}{c^2} j'_x$$

$$\boxed{j'_x = -\gamma \rho v} \quad (54)$$

So, the x component of Ampère-Maxwell law gives

$$\partial_y B'_z - \partial_z B'_y = -\frac{4\pi}{c}\gamma\rho v + \gamma \partial_0 E'_x + \gamma \frac{v}{c}\partial_x E'_x$$

$$\frac{v}{c}\left(\partial_y B'_z - \partial_z B'_y\right) = -\frac{4\pi}{c^2}\gamma v^2 \rho + \underbrace{\frac{\gamma v}{c}\partial_0 E'_x} + \frac{\gamma v^2}{c^2}\partial_x E'_x$$

Plugging the time derivative term into the transformed Gauss law:

$$\frac{v}{c}\left(\partial_y B'_z - \partial_z B'_y\right) + \frac{4\pi}{c^2}\gamma v^2 \rho - \frac{\gamma v^2}{c^2}\partial_x E'_x + \gamma \partial_x E'_x +$$

$$+ \partial_y E'_y + \partial_z E'_z = 4\pi\gamma\rho$$

$$\frac{1}{\gamma}\partial_x E'_x + \partial_y\left(E'_y + \frac{v}{c}B'_z\right) + \partial_z\left(E'_z - \frac{v}{c}B'_y\right) = \frac{4\pi}{\gamma}\rho$$

and we require

$$\begin{cases} \gamma\left(E'_y + \frac{v}{c}B'_z\right) = E_y \\ \gamma\left(E'_z - \frac{v}{c}B'_y\right) = E_z \end{cases}$$

This is not over yet. Let us find $B'_z$ and $B'_y$ in terms of the components of fields in the original frame.

For the Gauss law for the magnetic field,

$$\partial'_x B'_x + \partial'_y B'_y + \partial'_z B'_z = 0$$

Then

$$\frac{\gamma v}{c}\partial_0 B'_x + \gamma\partial_x B'_x + \partial_y B'_y + \partial_z B'_z = 0$$

and now, considering the Faraday law, we take the x component

$$\partial'_y E'_z - \partial'_z E'_y + \partial'_0 B'_x = 0$$

$$\partial_y E'_z - \partial_z E'_y + \gamma\partial_0 B'_x + \frac{\gamma v}{c}\partial_x B'_x = 0$$

Multiplying by $v/c$



$$\frac{v}{c}\left(\partial_y E'_z - \partial_z E'_y\right) + \frac{\gamma v}{c}\partial_0 B'_x + \gamma \frac{v^2}{c^2}\partial_x B'_x = 0$$

and we can take the time derivative term from here into the Gauss law for the magnetic field:

$$\frac{v}{c}\partial_z E'_y - \frac{v}{c}\partial_y E'_z - \frac{\gamma v^2}{c^2}\partial_x B'_x + \gamma \partial_x B'_x + \partial_y B'_y + \partial_z B'_z = 0$$

$$\frac{1}{\gamma}\partial_x B'_x + \partial_y\left(B'_y - \frac{v}{c}E'_z\right) + \partial_z\left(B'_z + \frac{v}{c}E'_y\right) = 0$$

Then we take

$$\begin{cases} B'_x = B_x \\ \gamma\left(B'_y - \frac{v}{c}E'_z\right) = B_y \\ \gamma\left(B'_z + \frac{v}{c}E'_y\right) = B_z \end{cases}$$

Finally, we can combine all these transformations to get

$$E'_y + \frac{v}{c}B'_z = \frac{1}{\gamma}E_y \qquad B'_z + \frac{v}{c}E'_y = \frac{1}{\gamma}B_z$$

$$E'_y + \frac{v}{c}\left(\frac{1}{\gamma}B_z - \frac{v}{c}E'_y\right) = \frac{1}{\gamma}E_y$$

$$E'_y\left(1 - \left(\frac{v}{c}\right)^2\right) = \frac{1}{\gamma}E_y - \frac{1}{\gamma}\frac{v}{c}B_z$$

$$\boxed{E'_y = \gamma\left(E_y - \frac{v}{c}B_z\right)}$$

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$$E'_z = \frac{E_z}{\gamma} - \frac{v}{c} B'_y \, , \quad B'_y = \frac{B_y}{\gamma} - \frac{v}{c} E'_z$$

$$E'_z = \frac{E_z}{\gamma} - \frac{v}{c}\left( \frac{B_y}{\gamma} - \frac{v}{c} E'_z \right)$$

$$E'_z\left(1 - \left(\frac{v}{c}\right)^2\right) = \frac{1}{\gamma}\left( E_z - \frac{v}{c} B_y \right)$$

$$\boxed{E'_z = \gamma\left( E_z - \frac{v}{c} B_y \right)}$$

Then

$$B'_y = \frac{B_y}{\gamma} - \frac{v}{c}\gamma\left( E_z - \frac{v}{c} B_y \right) = -\frac{v}{c}\gamma E_z + \underbrace{\frac{1}{\gamma}\left(1 + \gamma^2 \frac{v^2}{c^2}\right)}_{\gamma} B_y$$

$$\boxed{B'_y = \gamma\left( B_y - \frac{v}{c} E_z \right)}$$

$$B'_z = \frac{B_z}{\gamma} - \frac{v}{c}\gamma\left(E_y - \frac{v}{c} B_z\right) = \frac{B_z}{\gamma}\left(1 + \gamma^2\frac{v^2}{c^2}\right) - \gamma\frac{v}{c} E_y$$

$$\boxed{B'_z = \gamma\left(B_z - \frac{v}{c} E_y\right)}$$

It is clear now that $\vec{E}$ and $\vec{B}$ are not $SO(1,3)$ vectors, i.e., they are not Lorentz tensors.

In fact, these transformations suggest $E^i$ and $B^i$ are components of a Lorentz tensor.

Because we are in 4 dimensional space, the only way to accomodate 6 components into a tensor is with an antisymmetric 4×4 matrix:

$$\begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{55}$$

or, equivalently

$$\begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{56}$$

Moreover, we also expect the charge density and current to be defined in terms of a Lorentz tensor, so, we may take these 4 degrees of freedom into a 4-vector

$$J^\mu = \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \tag{57}$$

Let us define $F_{0i} = E_i$ and $F_{ij} = -\epsilon_{ijk} B_k$.
Moreover, the $2^{nd}$ possible tensor we mentioned is $\tilde{F}_{0i} = B_i$ and $\tilde{F}_{ij} = \epsilon_{ijk} E_k$.

The Maxwell equations are of first order in these fields, so we have to consider things like $\partial_\mu F^{\mu\nu}$ and $\partial_\mu \tilde{F}^{\mu\nu}$.

In the first case,

$$\partial_\mu F^{\mu\nu} = \begin{pmatrix} \partial_j F^{j0} \\ \partial_0 F^{0i} + \partial_j F^{ji} \end{pmatrix} = \begin{pmatrix} \partial_j E_j \\ -\frac{1}{c}\frac{\partial}{\partial t} E_i - \epsilon_{jik} \partial_j B_k \end{pmatrix}$$

$$= \begin{pmatrix} 4\pi\rho \\ -\frac{4\pi}{c}\vec{j} \end{pmatrix} = \frac{4\pi}{c}\begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix} = \frac{4\pi}{c} J^\nu$$

and in the second,

$$\partial_\mu \tilde{F}^{\mu\nu} = \begin{pmatrix} \partial_j \tilde{F}^{j0} \\ \partial_0 \tilde{F}^{0i} + \partial_j \tilde{F}^{ji} \end{pmatrix} = \begin{pmatrix} \partial_j B_j \\ -\frac{1}{c}\frac{\partial}{\partial t} B_i + \epsilon_{jik} \partial_j E_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vec{0} \end{pmatrix}$$

Thus,

$$\begin{cases} \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu & (58) \\ \partial_\mu \tilde{F}^{\mu\nu} = 0 & (59) \end{cases}$$